Linear waves in relativistic anisotropic magnetohydrodynamics

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Linear waves are investigated in the framework of the relativistic anisotropic magnetohydrodynamics in a fully invariant formulation. Dispersion relations are derived for the relativistic analog of slow, intermediate, and fast waves in a plasma with an anisotropic stress tensor. Relativistic Chew-Goldberger-Low dispersion relations are obtained also.

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I. INTRODUCTION

Properties of relativistic plasmas embedded in a strong external magnetic field are of considerable interest primarily because of possible applications in various astrophysical objects, such as pulsar winds and relativistic jets. The slow and large-space-scale motion of such plasmas is usually described within the framework of relativistic magnetohydrodynamics (MHD) [1,2] for a single fluid with isotropic pressure. However, in a number of cases the pressure apparently is not isotropic, since strong magnetic fields suppress energy exchange among parallel and perpendicular degrees of freedom in collisionless plasmas. In this case MHD should be generalized to include anisotropic pressure, in the spirit of the well-known Chew-Goldberger-Low (CGL) theory [3]. In some cases the very validity of the MHD approach is questionable [4].

MHD equations for a single relativistic fluid with anisotropic pressure were derived in [5] from the relativistic Vlasov equation in a manifestly invariant way. In the present paper we analyze properties of linear waves within the framework of the obtained relativistic anisotropic magnetohydrodynamics (RAM). The paper's organization is as follows. In Sec. II the RAM basic equations are given and their derivation method is described briefly. In Sec. III the RAM general dispersion relations are derived. In Sec. IV various limits are considered.

II. BASIC EQUATIONS

We consider a collisionless plasma that is embedded in the magnetic field. Both plasma fluid velocity and temperature are assumed to be relativistic. Hereafter we will use units where light velocity $c = 1$. Let U^α be the plasma 4-velocity and $F^{\alpha\beta}$ be the electromagnetic-field tensor. The plasma rest-frame electric and magnetic fields are defined as follows:

$$E^\alpha = F^{\alpha\beta} U_\beta \,, \tag{1}$$

$$B^\alpha = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} U_\beta F_{\gamma\delta} \,, \tag{2}$$

so that the electromagnetic field tensor can be written in the following form:

$$F^{\alpha\beta} = (E^\alpha U^\beta - E^\beta U^\alpha) + \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(B_\gamma U_\delta - B_\delta U_\gamma) \,, \tag{3}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is a completely antisymmetric tensor and $\epsilon^{0123} = 1$.

We apply the usual MHD frozen-in condition $E^\alpha \equiv 0$. The RAM equations can be written in the following form [5]:

$$J^\alpha_{,\alpha} = 0 \,, \tag{4}$$

$$^*F^{\alpha\beta}_{,\beta} = 0 \,, \tag{5}$$

$$T^{\alpha\beta}_{,\beta} = 0 \,, \tag{6}$$

$$J^\alpha = \rho U^\alpha \,, \tag{7}$$

$$^*F^{\alpha\beta} = B(n^\alpha U^\beta - n^\beta U^\alpha) \,, \tag{8}$$

$$T^{\alpha\beta} = W_1 U^\alpha U^\beta - W_2 g^{\alpha\beta} - W_3 n^\alpha n^\beta \,, \tag{9}$$

where $B^\alpha = B n^\alpha$ and $n^\alpha n_\alpha = -1$.

These expressions are quite general. The quantities ρ, U^α, and $T^{\alpha\beta}$ are related to the distribution function $f(u^\alpha)$. We assume that the magnetic field B is strong, i.e., $\Omega T \gg 1$, $\Omega L \gg 1$, where $\Omega = eB/M$ is the ion cyclotron frequency and T and L are typical temporal and spatial scales of the plasma motion. Then in the zeroth-order approximation $f = f_0(u_\parallel, u_\perp^2)\theta(u_0)\delta(u_\alpha u^\alpha - 1)$, where $u_0 = u^\alpha U_\alpha$, $u_\parallel = u^\alpha n_\alpha$, and $u_\alpha u^\alpha = u_0^2 - u_\parallel^2 - u_\perp^2 = 1$. (For convenience we also set $M = 1$, so that one does not have to distinguish between u^α and $p^\alpha = M u^\alpha$.) In this case

$$W_1 = \varepsilon + p_\perp + \frac{B^2}{4\pi} \,, \tag{10}$$

$$W_2 = p_\perp + \frac{B^2}{8\pi} \,, \tag{11}$$

$$W_3 = p_\parallel - p_\perp - \frac{B^2}{4\pi} \,, \tag{12}$$

where $\rho = \langle u_0 \rangle$, $\varepsilon = \langle u_0^2 \rangle$, $p_\parallel = \langle u_\parallel^2 \rangle$, $p_\perp = \langle u_\perp^2/2 \rangle$, and the averaging is defined as follows:

$$\langle X \rangle = \int du_\parallel \, u_\perp du_\perp \, u_0^{-1} \, X \, f_0(u_\parallel, u_\perp^2) \,. \tag{13}$$

The motion equations should be completed with the corresponding state equations, which in this case take the following form:

$$\varepsilon = \rho e(\rho, B) \tag{14}$$

$$p_\parallel = \rho^2 \frac{\partial e}{\partial \rho} = \frac{\partial \varepsilon}{\partial \ln \rho} - \varepsilon \,, \tag{15}$$

$$p_\perp = p_\parallel + \rho B \frac{\partial e}{\partial B} = \frac{\partial \varepsilon}{\partial \ln \rho} + \frac{\partial \varepsilon}{\partial \ln B} - \varepsilon \,. \tag{16}$$

These state equations represent anisotropic generalization of the isotropic state equation $p = p(\rho)$ and are more general than the usual CGL state equations [3] $p_\perp \propto \rho B$, $p_\parallel \propto \rho^3/B^2$. It is of interest to note that the quantities $q_\perp = \langle u_0 u_\perp^2/2 \rangle$ and $q_\parallel = \langle u_0 u_\parallel^2 \rangle$ ("modified pressure," according to [6]) satisfy the CGL relations: $q_\perp/\rho B = \text{const}$ and $q_\parallel B^2/\rho^3 = \text{const}$.

III. RAM DISPERSION RELATIONS

For linear wave analysis we introduce small perturbations in the form $B \to B + \delta B$, $\rho \to \rho + \delta\rho$, $U^\alpha \to U^\alpha + \delta U^\alpha$, $n^\alpha \to n^\alpha + \delta n^\alpha$, and assume that $\delta B, \delta\rho, \delta U^\alpha, \delta n^\alpha \propto \exp(ik_\alpha x^\alpha)$. It should be noted, that because of $U_\alpha U^\alpha = 1$, $n_\alpha n^\alpha = -1$, one has the following constraints: $U^\alpha \delta U_\alpha = n^\alpha \delta n_\alpha = 0$. Substituting in (4)–(6) $\partial/\partial x^\alpha \to ik_\alpha$, one obtains equations for perturbations in the following form:

$$k_\alpha \delta J^\alpha = 0 \,, \tag{17}$$

$$k_\alpha \delta^* F^{\alpha\beta} = 0 \,, \tag{18}$$

$$k_\alpha T^{\alpha\beta} = 0 \,, \tag{19}$$

where

$$\delta J^\alpha = \delta\rho U^\alpha + \rho \delta U^\alpha \,, \tag{20}$$

$$\begin{aligned} \delta^* F^{\alpha\beta} = {} & \delta B(n^\alpha U^\beta - n^\beta U^\alpha) + B(\delta n^\alpha U^\beta - \delta n^\beta U^\alpha) \\ & + B(n^\alpha \delta U^\beta - n^\beta \delta U^\alpha) \,, \end{aligned} \tag{21}$$

$$\begin{aligned} \delta T^{\alpha\beta} = {} & \delta W_1 U^\alpha U^\beta + W_1(\delta U^\alpha U^\beta + U^\alpha \delta U^\beta) \\ & - \delta W_2 g^{\alpha\beta} - \delta W_3 n^\alpha n^\beta \\ & - W_3(\delta n^\alpha n^\beta + n^\alpha \delta n^\beta) \,. \end{aligned} \tag{22}$$

Let us introduce the following notations:

$$k_\alpha U^\alpha = \omega \,, \qquad k_\alpha n^\alpha = k_\parallel \,, \tag{23}$$

$$k_\alpha k^\alpha = s^2 = \omega^2 - k_\parallel^2 - k^2 - k_\perp^2 \,, \tag{24}$$

where, obviously ω is the wave frequency in the plasma rest frame and $k_\parallel$ is the wave-vector component parallel to the magnetic field in the plasma rest frame. With these notations (17)–(19) take the following form:

$$\omega \delta\rho + \rho(k_\alpha \delta U^\alpha) = 0 \,, \tag{25}$$

$$\begin{aligned} & \delta B(k_\parallel U^\beta - \omega n^\beta) + B((k_\alpha \delta n^\alpha) U^\beta - \omega \delta n^\beta) \\ & \qquad + B(k_\parallel \delta U^\beta - (k_\alpha \delta U^\alpha) n^\beta) = 0 \,, \end{aligned} \tag{26}$$

$$\begin{aligned} & W_1(\omega \delta U^\beta + U^\beta (k_\alpha \delta U^\alpha)) + \omega \delta W_1 U^\beta \\ & \quad - k^\beta \delta W_2 - W_3((k_\alpha \delta n^\alpha) n^\beta + k_\parallel \delta n^\beta) - \delta W_3 k_\parallel n^\beta = 0 \,, \end{aligned} \tag{27}$$

and since $W_i = W_i(\rho, B)$, one has

$$\delta W_i = \frac{\partial W_i}{\partial \rho} \delta\rho + \frac{\partial W_i}{\partial B} \delta B \,.$$

One can simplify Eqs. (25)–(27) with the help of multiplication by four independent vectors k^α, U^α, n^α, $l^\alpha = \epsilon^{\alpha\beta\mu\nu} k_\beta U_\mu n_\nu$, $l^\alpha U_\alpha = l^\alpha n_\alpha = l^\alpha k_\alpha = 0$.

Multiplying (25)–(27) by l_α, one obtains

$$-\omega B(l_\alpha \delta n^\alpha) + B k_\parallel (l_\alpha \delta U^\alpha) = 0 \,, \tag{28}$$

$$W_1 \omega (l_\alpha \delta U^\alpha) - k_\parallel W_3 (l_\alpha \delta n^\alpha) = 0 \,, \tag{29}$$

and the corresponding dispersion relation for the RAM analog of the intermediate wave reads

$$\omega^2 = k_\parallel^2 W_3/W_1 = k_\parallel^2 \frac{p_\perp - p_\parallel + B^2/4\pi}{\varepsilon + p_\perp + B^2/4\pi} \,. \tag{30}$$

Multiplying by three other vectors, we obtain

$$k_\parallel \delta B + B(k_\alpha \delta n^\alpha) - \omega B(U_\alpha \delta n^\alpha) = 0 \,, \tag{31}$$

$$\omega \delta B + k_\parallel B(n_\alpha \delta U^\alpha) + B(k_\alpha \delta U^\alpha) = 0 \,, \tag{32}$$

$$\begin{aligned} & 2\omega W_1(k_\alpha \delta U^\alpha) + \omega^2 \delta W_1 - s^2 \delta W_2 - 2k_\parallel W_3 (k_\alpha \delta n^\alpha) \\ & \qquad - \delta W_3 k_\parallel^2 = 0 \,, \end{aligned} \tag{33}$$

$$W_1(k_\alpha \delta U^\alpha) + \omega \delta W_1 - \omega \delta W_2 - k_\parallel W_3 (U_\alpha \delta n^\alpha) = 0 \,, \tag{34}$$

$$\omega W_1 (n_\alpha \delta U^\alpha) - k_\parallel \delta W_2 + W_3 (k_\alpha \delta n^\alpha) + k_\parallel \delta W_3 = 0 \,, \tag{35}$$

together with

$$\rho(k_\alpha \delta U^\alpha) = -\omega\delta\rho\,. \quad (36)$$

The consistency condition for this set gives the dispersion relation for RAM analogs of the slow and fast magnetosonic waves in the following form:

$$\left(\omega^2 W_1 - k_\perp^2 \frac{\partial W_2}{\partial \ln B} - k_\parallel^2 W_3\right)\left(\omega^2 \frac{\partial (W_1 - W_2)}{\partial \ln \rho} - k_\parallel^2 \frac{\partial (W_2 - W_3)}{\partial \ln \rho}\right)$$

$$= k_\perp^2 \frac{\partial W_2}{\partial \ln \rho}\left[\omega^2\left(W_1 - \frac{\partial (W_1 - W_2)}{\partial \ln B}\right) + k_\parallel^2\left(W_3 + \frac{\partial (W_2 - W_3)}{\partial \ln B}\right)\right]\,. \quad (37)$$

Let us introduce the wave phase velocity $v^2 = \omega^2/k^2$ and angle of propagation θ with respect to the external magnetic field in the plasma rest frame, $\sin^2\theta = k_\perp^2/k^2$, $\cos^2\theta = k_\parallel^2/k^2$, and total wave number $k^2 = \omega^2 - s^2$. Then, introducing characteristic velocities as follows,

$$v_s^2 = \frac{\frac{\partial p_\parallel}{\partial \ln \rho}}{\varepsilon + p_\parallel}\,, \quad (38)$$

$$v_t^2 = \frac{\frac{\partial p_\perp}{\partial \ln \rho}}{\varepsilon + p_\parallel}\,, \quad (39)$$

$$v_A^2 = \frac{p_\perp - p_\parallel + (B^2/4\pi)}{\varepsilon + p_\perp + (B^2/4\pi)}\,, \quad (40)$$

$$v_F^2 = \frac{\frac{\partial p_\perp}{\partial \ln B} + \frac{\partial p_\perp}{\partial \ln \rho} + \frac{B^2}{4\pi}}{\varepsilon + p_\perp + \frac{B^2}{4\pi}}\,, \quad (41)$$

we write eventually the dispersion relation for the fast and slow waves in the following form:

$$v^4 - v^2\Phi((v_s^2 + v_A^2)\cos^2\theta + v_F^2 \sin^2\theta)$$
$$+\cos^2\theta(v_s^2(v_A^2\cos^2\theta + v_F^2\sin^2\theta) - v_t^4(1 - v_A^2)\sin^2\theta) = 0\,, \quad (42)$$

while the dispersion relation for the Alfvén (intermediate) wave looks as follows:

$$v^2 = v_A^2 \cos^2\theta\,. \quad (43)$$

In the isotropic pressure limit, $p_\parallel = p_\perp = p$ and $\partial\varepsilon/\partial \ln B = 0$. The corresponding dispersion relations take the following form:

$$v^2 = v_A^2\cos^2\theta\,, \quad (44)$$

$$v^4 - v^2(v_A^2 + v_s^2 - v_A^2 v_s^2 \sin^2\theta) + v_A^2 v_s^2 \cos^2\theta = 0\,, \quad (45)$$

where now

$$v_A^2 = \frac{B^2/4\pi}{\varepsilon + p + B^2/4\pi}\,. \quad (46)$$

In the nonrelativistic limit $v, v_A, v_s, v_t, v_F \ll 1$, $p_\parallel, p_\perp, B^2/4\pi \ll \varepsilon \approx \rho$, and the well-known MHD dispersion relations are immediately rederived.

IV. VARIOUS LIMITS

It is of interest to analyze various special cases of the dispersion relations (42)–(43).

(i) In the case of the parallel propagation $\theta = 0$ and one immediately has

$$v_1^2 = v_2^2 = v_A^2, \qquad v_3^2 = v_s^2\,. \quad (47)$$

Aperiodic instability occurs when $p_\parallel > p_\perp + (B^2/4\pi)$.

(ii) In the case of the perpendicular propagation $\theta = \pi/2$ one obtains

$$v_1^2 = v_F^2, \qquad v_2^2 = v_A^2, \qquad v_3^2 = 0\,. \quad (48)$$

The instability criterion is the same as above.

(iii) In the case of the perpendicularly cold plasma $p_\perp = 0$ one finds $\varepsilon = \rho e(\rho/B)$ and $v_t = 0$, and it is easy to obtain

$$v_1^2 = \frac{(B^2/4\pi) - p_\parallel \cos^2\theta}{\varepsilon + B^2/4\pi}\,, \quad (49)$$

$$v_2^2 = \frac{(B^2/4\pi) - p_\parallel}{\varepsilon + B^2/4\pi}\,, \quad (50)$$

$$v_3^2 = v_s^2\cos^2\theta\,. \quad (51)$$

Aperiodic instability occurs when $p_\parallel > B^2/4\pi$.

(iv) In the case of the zero parallel temperature $p_\parallel = 0$ and $\varepsilon = \rho e(B)$, so that one has

$$\varepsilon\left(\varepsilon + p_\perp + \frac{B^2}{4\pi}\right)v^4 - v^2\left(p_\perp + \frac{B^2}{4\pi} + \frac{\partial p_\perp}{\partial \ln B}\sin^2\theta\right)$$
$$-p_\perp \sin^2\theta\cos^2\theta = 0\,, \quad (52)$$

and one of the roots is always negative. Thus, plasma with zero parallel temperature is always unstable.

(v) Chew-Goldberger-Law state equations read $p_\parallel B^2/\rho^3$ =const and $p_\perp B/\rho$ =const. This form of pressure implies the following functional form for the specific energy ε:

$$\varepsilon = \rho\left(1 + k_1\frac{\rho^2}{B^2} + k_2 B\right)\,, \quad (53)$$

where k_1 and k_2 are constants. In this case $\varepsilon = \rho + p_\| /2 + p_\perp$, $\partial p_\| / \partial \ln \rho = 3p_\|$, $\partial p_\perp / \partial \ln \rho = p_\perp$, $\partial p_\perp / \partial \ln B = -p_\perp$, $\partial p_\| / \partial \ln B = -2p_\|$, and it easy to obtain

$$v_s^2 = \frac{3p_\|}{\rho + p_\perp + 3p_\|/2}, \tag{54}$$

$$v_t^2 = \frac{p_\perp}{\rho + p_\perp + 3p_\|/2}, \tag{55}$$

$$v_A^2 = \frac{p_\perp - p_\| + (B^2/4\pi)}{\rho + 2p_\perp + p_\|/2 + (B^2/4\pi)}, \tag{56}$$

$$v_F^2 = \frac{2p_\perp + B^2/4\pi}{\rho + 2p_\perp + p_\|/2 + (B^2/4\pi)}. \tag{57}$$

The nonrelativistic limit is achieved when $p_\perp, p_\|, B^2/4\pi \ll \varepsilon \approx \rho$. In this case it is convenient to introduce typical nonrelativistic notation:

$$v_\|^2 = \frac{p_\|}{\rho}, \qquad v_\perp^2 = \frac{p_\perp}{\rho}, \qquad V_A^2 = \frac{B^2}{4\pi\rho}, \tag{58}$$

where $v_\|$ and $v_\perp$ are the parallel and perpendicular thermal velocities, respectively.

The resulting dispersion relation is

$$v^4 - v^2[(V_A^2 + 2v_\perp^2) + (2v_\|^2 - v_\perp^2)\cos^2\theta]$$
$$+ \cos^2\theta[3v_\|^2(V_A^2 + 2v_\perp^2) - v_\perp^4]$$
$$+[3v_\|^4 + v_\perp^2(3v_\|^2 - v_\perp^2)\cos^2\theta] = 0, \tag{59}$$

which coincides with the well-known nonrelativistic CGL dispersion relation [8].

V. CONCLUSIONS

We have derived the analog for the MHD waves dispersion relations in a relativistic anisotropic plasma. The obtained dispersion relations are reduced to the well-known nonrelativistic relations in the isotropic case and in the case of the relativistic CGL functional form.

The applications of the obtained RAM dispersion relations are primarily to astrophysical systems with relativistic plasmas. One such system is relativistic electron-positron pulsar wind. It is widely believed that in the inner zone the plasma in this wind is perpendicularly cold, $p_\perp = 0$, and the magnetic pressure dominates, $B^2/4\pi \gg p_\|$. If the wind is spherically symmetric [8], then $\rho \propto B \propto 1/r^2$ in the close zone where the poloidal field dominates, or $\rho \propto B \propto 1/r$ in the far zone where the toroidal field dominates. In any case ρ/B =const and $B \to 0$ as the distance $r \to \infty$. Therefore, if the relativistic CGL holds,

$$\frac{p_\|}{B^2} \propto \left(\frac{\rho^2}{B^2}\right)\left(\frac{\partial e}{\partial(\rho/B)}\right)\left(\frac{1}{B}\right)$$

and becomes large at large distances. According to (50) the wind becomes unstable. The developing fire-hose instability should result in effective pressure isotropy.

[1] F. Hoffman and E. Teller, Phys. Rev. **80**, 692 (1950).
[2] A. Lichnerowitz, *Relativistic Hydrodynamics and Magnetohydrodynamics* (Benjamin, New York, 1967).
[3] G. Chew, M. Goldberger, and F. Low, Proc. R. Soc. London, Ser. A **236**, 1204 (1954).
[4] C.F. Kennel, M.E. Gedalin, and J.G. Lominadze, in *Plasma Astrophysics*, edited by T.D. Guyenne (European Space Agency, Paris, 1988), p. 137.
[5] M. Gedalin, Phys. Fluids B **3**, 1871 (1991).
[6] E.G. Tsikarishvili, J.G. Lominadze, A.D. Rogava, and I.I. Javakhishvili, Phys. Rev. A **46**, 1078 (1992).
[7] T.F. Volkov, Rev. Plasma Phys. **4**, 3 (1966).
[8] C.F. Kennel, F.S. Fujimura, and I. Okamoto, Geophys. Astrophys. Fluid Dyn. **26**, 147 (1983).